Applications of Linear Algebra

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1. In a town with n people, m clubs have been formed. If each club have an odd number of members and any two clubs have even number of members in common, then find the maximum number of clubs that can be formed.

2. Find maximum number of points in $\mathbb{R}^n$ such that the size of the set of distances between these points is atmost 2.

3. There are no four points in the plane such that distance between each pair is an odd integer.

4. Let m_{ij} for $i, j \in \{0, 1, \ldots, n\}$ be nonnegative real numbers such that $m_{ii} = 0$ for each i and $m_{ij} = m_{ji}$ for all $i \neq j$. Find a condition on these numbers m_{ij} for the existence of points $P_0, P_1, \ldots P_n$ with $\|P_i - P_j\| = m_{ij}$. (For example, if $n = 2$, the condition is *triangle inequality*).

5. Let S be a set of n elements and $M_1, M_2, \ldots, M_{n+1}$ be subsets of S. Show that there exist integers $r, s \geq 1$ and a disjoint sets of indices $\{i_1, i_2, \ldots, i_r\}$ and $\{j_1, j_2, \ldots, j_s\}$ such that $\bigcup_{k=1}^{r} M_{i_k} = \bigcup_{k=1}^{s} M_{j_k}$

6. Find the cubic equation, the zeros of which are the cubes of the roots of the euation $x^3 + ax^2 + bx + c = 0$.

7. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that $1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$ holds identicaly ?

8. Let S be a fnite set with n elements and and let $F = \{A_1, A_2, \ldots, A_n\}$ be a family of n distinct subsets of S. Prove that there exists an element $x \in S$ such that $A_1 \bigcup\{x\}, A_2 \bigcup\{x\}, \ldots, A_n \bigcup\{x\}$ are distinct.

9. Let n be an odd number and A be an $n \times n$ matrix whose entries are from the set $\{1, -1\}$. If the product of the entries of the i^{th} row is a_i and the product of the entries of the j^{th} column is b_j, prove that

$$\sum_{i=1}^{n} a_i + \sum_{j=1}^{n} b_j \neq 0.$$

Does the above assertion hold for an even n ?